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The Gabriel–Roiter measure

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Abstract

The first Brauer–Thrall conjecture asserts that algebras of bounded representation type have finite representation type. This conjecture was solved by Roiter in 1968. The induction scheme which he used in his proof prompted Gabriel to introduce an invariant which we propose to call Gabriel–Roiter measure. This invariant is defined for any finite length module and it will be studied in detail in this paper. Whereas Roiter and Gabriel were dealing with algebras of bounded representation type only, it is the purpose of the present paper to demonstrate the relevance of the Gabriel–Roiter measure for algebras in general, in particular for those of infinite representation type.

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Let Λ be an artin algebra (thus Λ is a ring, its center is artinian and Λ is finitely generated as a module over its center), we always may (and will) assume that Λ is connected (the center is a local ring). Let $\operatorname{Mod}\Lambda$ denote the category of all (left) Λ-modules and $\operatorname{mod}\Lambda$ the full subcategory of all finitely generated modules.

Usually, we will deal with finitely generated modules and call them just *modules*, given such a module M, we denote by $|M|$ its length. The paper deals with an invariant attached to such a module which has been introduced by Gabriel (under the name "Roiter measure") in order to explain Roiter's induction scheme in his solution of the first Brauer–Thrall con-

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jecture. Most of the results presented here are inspired by the methods used by Roiter in his proof, and by Gabriel's reformulations. Recall that the first Brauer–Thrall conjecture asserts that if Λ is of bounded representation type (this means that there is a bound on the length of the indecomposable representations), then Λ is of finite representation type (meaning that there are only finitely many isomorphism classes of indecomposable modules). Both Roiter and Gabriel have assumed from the beginning that Λ is of bounded representation type, however this assumption is really misleading: a proper reading of these papers shows that the methods exhibited by Roiter and the invariant introduced by Gabriel shed light on the structure of the category of Λ-modules for an arbitrary artin algebra Λ, especially for Λ of infinite representation type!

The Gabriel–Roiter measure $\mu(M)$ of a Λ-module M (as we propose to call this invariant) may be considered as a rational number (say between 0 and 1) which only depends on the submodule lattice of M. The main property is the following: the class of modules which are direct sums of modules M with $\mu(M) \leqslant r$ for a fixed real number r is closed under submodules. In this way, one obtains an interesting filtration of the category of all Λ-modules by subcategories which are closed under submodules.

We say that r is a Gabriel–Roiter measure for Λ, provided there are indecomposable Λ-modules M with $\mu(M) = r$. We will show that for Λ of infinite representation type, there are Gabriel–Roiter measures r_t, r^t for Λ with

$$r_1 < r_2 < r_3 < \cdots < r^3 < r^2 < r^1$$

such that any other Gabriel–Roiter measure r for Λ satisfies $r_t < r < r^t$ for all $t \in \mathbb{N}_1$. Also, for any t, there are only finitely many isomorphism classes of indecomposable modules with Gabriel–Roiter measure r_t or r^t. Note that any infinite set of Gabriel–Roiter measures for Λ provides arbitrarily large indecomposable modules, thus, in particular, we encounter in this way two different proofs of the first Brauer–Thrall conjecture.

We will say that the indecomposable modules with Gabriel–Roiter measure of the form r_t form the take-off part, those with Gabriel–Roiter measure of the form r^t the landing part of the category. We get in this way a partition of the module category into three parts: the take-off part, the central part and the landing part, where both the take-off part and the landing part are in some sense of combinatorial nature. The paper will provide additional information on the take-off and the landing modules.

1. Definition and main properties

Let $\mathbb{N}_1 = \{1, 2, \ldots\}$ be the set of natural numbers. Note that we use the symbol $\subset$ to denote proper inclusions. Let $\mathcal{P}(\mathbb{N}_1)$ be the set of all subsets $I \subseteq \mathbb{N}_1$. We consider this set as a *totally ordered* set as follows: If I, J are different subsets of $\mathbb{N}_1$, write $I < J$ provided the smallest element in $(I \setminus J) \cup (J \setminus I)$ belongs to J. (The subsets of $\mathbb{N}_1$ to be considered usually will be finite ones; for a visualization of this totally ordering on the subset $\mathcal{P}_f(\mathbb{N})$ of all finite subsets of $\mathbb{N}_1$, we refer to Appendix B.) Also, write $I \ll J$ provided $I \subset J$ and for all elements $a \in I, b \in J \setminus I$, we have $a < b$. We say that J *starts* with I provided $I = J$ or $I \ll J$.

- The totally ordered set $\mathcal{P}(\mathbb{N})$ is complete.

- If $I \subseteq J \subseteq \mathbb{N}_1$, then $I \leqslant J$.
- If $I_1 \leqslant I_2 \leqslant I_3$ and I_3 starts with I_1, then I_2 starts with I_1.

For a (not necessarily finitely generated) Λ-module M, let $\mu(M)$ be the supremum of the sets $\{|M_1|, \ldots, |M_t|\}$, where $M_1 \subset M_2 \subset \cdots \subset M_t$ is a chain of indecomposable submodules of M, we call $\mu(M)$ the *Gabriel–Roiter measure* of M. In case there exists a chain of submodules (the index set being finite or countable)

$$M_1 \subset M_2 \subset \cdots \subseteq \bigcup_i M_i = M \quad \text{such that} \quad \mu(M) = \bigl\{|M_i| \mid i\bigr\},$$

then we call this chain a *Gabriel–Roiter filtration* of M. Note that *a finitely generated Λ-module M has a Gabriel–Roiter filtration if and only if M is indecomposable.* We will see below that an infinitely generated module with a Gabriel–Roiter filtration is indecomposable (Theorem 1), but conversely there are indecomposable infinitely generated modules without a Gabriel–Roiter filtration (of course, any module with a Gabriel–Roiter filtration is countably generated, an example of a countable generated indecomposable module without a Gabriel–Roiter filtration will be given in Appendix B).

Finally, we call an inclusion $N \subset M$ of finitely generated indecomposable Λ-modules a *Gabriel–Roiter inclusion* provided $\mu(M) = \mu(N) \cup \{|M|\}$, thus if and only if every proper submodule of M has Gabriel–Roiter measure at most $\mu(N)$. Note that *a chain $M_1 \subset M_2 \subset \cdots \subseteq \bigcup_i M_i = M$ is a Gabriel–Roiter filtration if and only if all the inclusions $M_i \subset M_{i+1}$ are Gabriel–Roiter inclusions.*

- If M' is a submodule of M, then $\mu(M') \leqslant \mu(M)$.
- For any module M, the Gabriel–Roiter measure $\mu(M)$ is the supremum of $\mu(M')$, where M' is a finitely generated indecomposable submodule of M.

Main property (Gabriel). *Let $X, Y_1, \ldots, Y_t$ be indecomposable Λ-modules of finite length and assume that there is a monomorphism $f : X \to \bigoplus_{i=1}^t Y_i$.*

(a) *Then $\mu(X) \leqslant \max \mu(Y_i)$.*
(b) *If $\mu(X) = \max \mu(Y_i)$, then f splits.*
(c) *If $\max \mu(Y_i)$ starts with $\mu(X)$, then there is some j such that $\pi_j f$ is injective, where $\pi_j : \bigoplus_i Y_i \to Y_j$ is the canonical projection.*

Parts (a) and (b) have been formulated and proven by Gabriel in [4] (using the additional assumption that Λ is of bounded representation type, with a footnote that this assumption may be deleted).

Proof. We write $Y = \bigoplus_i Y_i$ and $\mu_Y = \max \mu(Y_i)$ (it will follow from the main property that $\mu_Y = \mu(\bigoplus Y_i)$, see Corollary 1, but at this stage, this is not known). Denote by $\pi_j : Y = \bigoplus_i Y_i \to Y_j$ the projection maps. We are going to show (a) and (c) by induction. Note that (c) immediately implies (b): namely, if we have equality $\mu(X) = \mu_Y$, then according to (c) we find an index j such that the composition of f and the projection $\pi_j : Y \to Y_j$ is injective. An injective map $X \to Y_j$ implies that $\mu(X) \leqslant \mu(Y_j)$;

together with $\mu(Y_j) \leqslant \mu_Y = \mu(X)$ we see that $\mu(X) = \mu(Y_j)$, in particular $|X| = |Y_j|$. As a consequence, the injective map $\pi_j f : X \to Y_j$ has to be bijective, thus f is a split monomorphism.

For the proof of (a) and (c), we use induction on $t = |X| + |Y|$, starting with trivial cases, say $t = 0$ or $t = 1$. First, we claim that we can assume that $\pi_i f$ is surjective, for all i. If not, then let $Y'_i = \pi_i f(X)$, this is a submodule of Y_i. Decompose $Y'_i = \bigoplus_j Y_{ij}$ with indecomposable modules Y_{ij}. Note that X embeds under f into $\bigoplus_i Y'_i = \bigoplus_{ij} Y_{ij}$, and $|X| + |Y| > |X| + \sum_{ij} |Y_{ij}|$, thus by induction we know the first of the following two inequality signs:

$$\mu(X) \leqslant \max \mu(Y_{ij}) \leqslant \mu_Y,$$

the second inequality sign is due to the fact that any Y_{ij} is a submodule of Y_i. This shows (a). Also, assume μ_Y starts with $\mu(X)$, then we see that $\max \mu(Y_{ij})$ starts with $\mu(X)$ and therefore by induction we know that there is some projection map $\pi_{rs} : \bigoplus_{ij} Y_{ij} \to Y_{rs}$ such that the composition of the inclusion $f' : X \to \bigoplus_{ij} Y_{ij}$ with π_{rs} is a monomorphism. Now, the projection π_{rs} is the composition of first the projection $\pi'_r : \bigoplus_{ij} Y_{ij} = \bigoplus_i Y'_i \to Y'_r$ and then the projection $Y'_r = \bigoplus_j Y_{rj} \to Y_{rs}$. It follows that $\pi'_r f' : X \to Y'_r$ is a monomorphism. If we compose this monomorphism with the embedding $Y'_r \to Y_r$, we just obtain $\pi_r f$. This shows (c).

Next, note that we even may assume that $\pi_i f$ is a proper epimorphism, for all i. Namely, in case some $\pi_i f$ is an isomorphism, then X is a direct summand of Y_i, and since Y_i is indecomposable, X is isomorphic to Y_i. Thus, (a) is satisfied, but of course also (c).

Since we assume now that $\pi_i f$ is a proper epimorphism, for all i, we know that $|X| > |Y_i|$, for all i. In particular, X cannot be simple, since otherwise f would be an embedding of the non-zero module X into the zero module. In particular, there is an indecomposable submodule X' of X such that the embedding $g : X' \to X$ is a Gabriel–Roiter inclusion. Now apply the induction to the embedding $gf : X' \to Y$. It follows that $\mu(X') \leqslant \mu_Y$, according to (a).

The case $\mu(X') = \mu_Y$ can be ruled out for the following reason: Since (c) implies (b), the inclusion map $X' \subseteq Y$ would be a split monomorphism. However, this map factors through the inclusion map $X' \subset X$, thus also $X' \subset X$ would be a split monomorphism. However X is indecomposable.

Now let us recall what it means that $\mu(X') < \mu(Y_j)$: there is a natural number a which belongs to $\mu(Y_j) \setminus \mu(X')$ and $\mu(X') \cap [1, a-1] = \mu(Y_j) \cap [1, a-1]$. Since a does not belong to $\mu(X')$, we have in particular $a \neq |X'|$. Now, $a \leqslant |Y_j| < |X|$, thus a does not belong to $\mu(X) = \mu(X') \cup \{|X|\}$ and therefore a belongs to $\mu(Y_j) \setminus \mu(X)$. Second, again using that $a < |X|$ and that $\mu(X) = \mu(X') \cup \{|X|\}$, we also see that $\mu(X) \cap [1, a-1] = \mu(X') \cap [1, a-1]$, and therefore $\mu(X) \cap [1, a-1] = \mu(Y_j) \cap [1, a-1]$. Both assertions together yield $\mu(X) < \mu(Y_j)$, thus (a).

For the proof of (c), we only have to observe that our assumption $|X| > |Y_i|$ for all i excludes that μ_Y may start with $\mu(X)$. Namely, $\mu_Y = \mu(Y_r)$ for some r, and if $\mu(Y_r)$ starts with $\mu(X)$, then in particular $|X| \leqslant |Y_r|$. This completes the induction step. □

Corollary 1. *If $M_1, \ldots, M_t$ are (not necessarily finitely generated) indecomposable Λ-modules, then $\mu(\bigoplus M_i) = \max \mu(M_i)$.*

Proof. Since M_i is a submodule of $M = \bigoplus M_i$, we have $\max \mu(M_i) \leqslant \mu(\bigoplus M_i)$. Conversely, $\mu(M)$ is the supremum of $\mu(M')$, where M' is a finitely generated indecomposable submodule of M, thus we have to show $\mu(M') \leqslant \max \mu(M_i)$. Now $M' \subseteq \bigoplus M'_i$, where M'_i is a finitely generated submodule of M_i. We can write $M'_i = \bigoplus_j M_{ij}$ with indecomposable modules M_{ij}. Note that M_{ij} is a submodule of M_i, thus $\mu(M_{ij}) \leqslant \mu(M_i)$. According to part (a) of the main property, we get $\mu(M') \leqslant \max_{ij} \mu(M_{ij}) \leqslant \max_i \mu(M_i)$, this concludes the proof. □

Corollary 2. *Let $N \subset M$ be a Gabriel–Roiter inclusion, and $f : N \to M$ an injective map. Then for any factorization $f = f'' f'$, where $f'' : N' \to M$ is a proper monomorphism, the map $f' : N \to N'$ is a split monomorphism.*

Actually, it is sufficient to formulate the case of dealing with inclusion maps: *If $N \subset M$ is a Gabriel–Roiter inclusion and N' is a proper submodule of M containing N, then the embedding $N \subseteq N'$ splits.*

Proof of Corollary 2. Write $N' = \bigoplus_i N_i$ with indecomposable modules N_i. The main property (a) asserts that $\mu(N) \leqslant \max \mu(N_i)$ and trivially $\max \mu(N_i) \leqslant \mu(M)$. The main property (b) asserts that f' is a split monomorphism. □

Corollary 3. *Assume $N \subset M$ is a Gabriel–Roiter inclusion. Then M/N is indecomposable.*

Note that the fact that an embedding $N \subset M$ is a Gabriel–Roiter inclusion depends only on the isomorphism classes of N and M, thus we see: *If $N \subset M$ is a Gabriel–Roiter inclusion, then the cokernel of any monomorphism $f : N \to M$ is indecomposable.* One should be aware that there are plenty of pairs of modules N, M such that there do exist monomorphisms $f : N \to M$ both with indecomposable and with decomposable cokernels.

Proof of Corollary 3. Assume $M/N = Q_1 \oplus Q_2$ with non-zero modules Q_1, Q_2. For $i = 1, 2$, write $Q_i = N_i/N$, where $N \subset N_i \subset M$. According to Corollary 2, we find submodules N'_i of N_i such that $N_i = N \oplus N'_i$. Then $M = N \oplus N'_1 \oplus N'_2$, in contrast to the fact that M is indecomposable. □

Theorem 1. *Any module M with a Gabriel–Roiter filtration is indecomposable.*

Proof. The case when M is finitely generated, is trivial. Thus, we can assume that there is given an infinite chain

$$M_1 \subset M_2 \subset \cdots \subseteq \bigcup_i M_i = M,$$

where all the inclusions $M_i \subset M_{i+1}$ are Gabriel–Roiter inclusions. Assume that there is given a direct decomposition $M = U \oplus V$ with U, V both non-zero. Note that if $U \cap M_i = 0$ for all i, then $U = U \cap M = U \cap (\bigcup M_i) = \bigcup (U \cap M_i) = 0$. This shows that there is some index s such that $U \cap M_s \neq 0$ and also $V \cap M_s \neq 0$. Choose finitely generated submodules

$U' \subseteq U$ and $V' \subseteq V$ such that $M_s \subseteq M' = U' \oplus V'$, and decompose $U' = \bigoplus U_i$, $V' = \bigoplus V_j$ with indecomposable modules U_i and V_j. Finally, choose t such that $M' \subseteq M_t$.

Now we consider the Gabriel–Roiter measures: We get

$$\mu(M_s) \leqslant \max\{\mu(U_i), \mu(V_j)\} \leqslant \mu(M_t)$$

(the first inequality is the main property (a), the second is trivial). Since M_s and M_t are connected by Gabriel–Roiter inclusions, $\mu(M_t)$ starts with $\mu(M_s)$, thus also $\max\{\mu(U_i), \mu(V_j)\}$ starts with $\mu(M_s)$ and we can apply the main property (b). Without loss of generality, we can assume that the composition of the inclusion $M_s \to \bigoplus_i U_i \oplus \bigoplus_j V_j = M'$ and the projection $\pi_i^U : M' \to U_1$ is injective (where $i = 1$ is one of the indices). Recall that there is a non-zero element $v \in V \cap M_s$. Since $M_s \subseteq M' = U' \oplus V'$, we can write $v = u' + v'$ with $u' \in U'$ and $v' \in V'$. However $u' = v - v' \in U' \cap V = 0$ shows that $v = v'$ belongs to V'. Since v belongs to $V' = \bigoplus V_j$, it is mapped under π_1^U to zero. This contradicts the fact that π_1^U is injective. □

2. Main results

As abbreviation, we write $\mathcal{A} = \operatorname{mod} \Lambda$. For any finite subset $I \subset \mathbb{N}_1$, we denote by $\mathcal{A}(I)$ the class of indecomposable Λ-modules M with $\mu(M) = I$. Similarly, let $\mathcal{A}(\leqslant I)$ be the class of indecomposable Λ-modules M with $\mu(M) \leqslant I$. According to the main property, $\operatorname{add} \mathcal{A}(\leqslant I)$ *is closed under submodules and any monomorphism* $f : X \to Y$ *with* X *in* $\mathcal{A}(I)$ *and* Y *in* $\operatorname{add} \mathcal{A}(\leqslant I)$ *splits.*

We say that I is a *Gabriel–Roiter measure for* Λ provided $\mathcal{A}(I)$ is non-empty. A Gabriel–Roiter measure I for Λ is said to be *of finite type* provided there are only finitely many isomorphism classes in $\mathcal{A}(I)$.

Note that the indecomposable Λ-modules of length at most n belong to the classes $\mathcal{A}(I)$ with $I \subseteq \{1, 2, \ldots, n\}$, and that there are just finitely many such classes. Thus Λ *is of unbounded representation type if and only if there are infinitely many Gabriel–Roiter measures for* Λ.

Recall that the first Brauer–Thrall conjecture has asserted that in case Λ is of infinite representation type, then Λ is of unbounded representation type. The conjecture has been shown in 1968 by Roiter [9] and the following result should be considered as a refinement of the paper.

Theorem 2. *Let* Λ *be of infinite representation type. Then there are Gabriel–Roiter measures* I_t, I^t *for* Λ (*with* $t \in \mathbb{N}_1$) *such that*

$$I_1 < I_2 < I_3 < \cdots < I^3 < I^2 < I^1$$

and such that any other Gabriel–Roiter measure I *for* Λ *satisfies* $I_t < I < I^t$ *for all* $t \in \mathbb{N}_1$. *Moreover, all these Gabriel–Roiter measures* I_t *and* I^t *are of finite type.*

We call the modules in $\bigcup_t \mathcal{A}(I_t)$ (or the additive category with these indecomposable modules) the *take-off part* of the category $\mathcal{A}$, and $\bigcup_t \mathcal{A}(I^t)$ (or the additive category with these indecomposable modules) the *landing part* of $\mathcal{A}$.

Note that *for any n, there are only finitely many isomorphism classes of indecomposable modules of length n which belong to the take-off part* (since they belong to only finitely many classes $\mathcal{A}(I_t)$ and any class $\mathcal{A}(I_t)$ is of finite type). Similarly, *there are also only finitely many isomorphism classes of indecomposable modules of length n which belong to the landing part.*

It is obvious that the modules in $\mathcal{A}(I_1)$ are just the simple modules, those in $\mathcal{A}(I_2)$ are the local modules of Loewy length 2 of largest possible length. On the other hand, the modules in $\mathcal{A}(I^1)$ are the indecomposable injective modules of largest possible length. For general t, it seems to be difficult to characterize the modules in $\mathcal{A}(I_t)$ or $\mathcal{A}(I^t)$ in a direct way.

Theorem 3. *Let Λ be of infinite representation type. There do exist modules which are not finitely generated and which have a Gabriel–Roiter filtration*

$$M_1 \subset M_2 \subset \cdots \subseteq \bigcup_i M_i = M$$

such that all the modules M_i belong to the take-off part.

Note that according to Theorem 1, such a module M is indecomposable. Also, any finitely generated submodule M' of M is contained in some M_t, thus belongs to the take-off part. In particular, for any natural number n, M has only finitely many isomorphism classes of submodules of length n.

The existence of infinitely generated indecomposable modules for any artin algebra of infinite representation type was first shown by Auslander [1]. For a discussion of the question whether a union of a chain of indecomposable modules of finite length is indecomposable or not, we refer to [5].

Recall that Auslander–Smalø have introduced in [3] the notion of preprojective and preinjective modules (actually with reference to the work of Roiter and Gabriel).

Theorem 4. *The modules in the landing part are preinjective.*

Since modules which have infinitely many different Gabriel–Roiter measures cannot have bounded length, we obtain in this way a new proof for the assertion that the indecomposable preinjective modules are of unbounded length [3, 5.11]. Two additional remarks should be useful. First, there usually will exist preinjective indecomposable modules which do not belong to the landing part. For example, any simple module belongs to $\mathcal{A}(I_1)$, thus a simple injective module is preinjective and in the take-off part, thus not in the landing part. But there may be even infinitely many isomorphism classes of preinjective indecomposable modules which do not belong to the landing part (see an example in Appendix B). Second, in contrast to Theorem 4, the modules in the take-off part are usually *not* preprojective. Note that in order to deal also with preprojective modules, we have to invoke the dual considerations, thus to work with a corresponding Gabriel–Roiter comeasure which is based on looking at indecomposable factor modules in contrast to the Gabriel–Roiter measure which is based on indecomposable submodules (see Appendix C).

The proofs of Theorems 2, 3 and 4 will be given in the following two sections. Section 3 will deal with the take-off part, Section 4 with the landing part of $\mathcal{A}$.

3. The take-off part

Before we construct the sets I_t and the corresponding classes $\mathcal{A}(I_t)$, we need some preparation. Note that the methods mentioned here when dealing with modules of finite length are all due to Roiter [9], we just present an elaboration. This concerns in particular the essential coamalgamation lemma and its use in order to show the finiteness of $\mathcal{A}(I_{t+1})$. It is very amazing that these consideration allow to bound the number of isomorphism classes with fixed length.

For any pair of finitely generated Λ-modules M, N, the groups $\mathrm{Hom}(M,N)$ and $\mathrm{Ext}^1(M,N)$ are k-modules of finite length, where k is the center of Λ, and we write $\dim \mathrm{Hom}(M,N)$ and $\dim \mathrm{Ext}^1(M,N)$ in order to denote the length as a k-module.

Ext-lemma. *Let N be an indecomposable module. Let*

$$0 \to N^n \oplus N' \xrightarrow{u} X \to Q \to 0$$

be an exact sequence. If $\dim \mathrm{Ext}^1(Q,N) < n$, then for at least one of the canonical inclusions $u_i : N \to N^n \oplus N'$, the composition uu_i is a split monomorphism.

This is a basic and well-known result in homological algebra.

Proof. Consider the projection maps $\pi_i : N^n \oplus N' \to N$ with $1 \leqslant i \neq n$ and the induced exact sequences ϵ_i (these are the lower exact sequences):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N^n \oplus N' & \xrightarrow{u} & X & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_i} & & \downarrow & & \| & & \\ 0 & \longrightarrow & N & \longrightarrow & X_i & \longrightarrow & Q & \longrightarrow & 0 \end{array} \qquad \epsilon_i$$

Since $\dim \mathrm{Ext}^1(Q,N) < n$, there is a non-trivial linear combination $\sum_i \lambda_i \epsilon_i = 0$. This means that the induced exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N^n \oplus N' & \xrightarrow{u} & X & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sum \lambda_i \pi_i} & & {\scriptstyle g}\downarrow & & \| & & \\ 0 & \longrightarrow & N & \longrightarrow & X' & \longrightarrow & Q & \longrightarrow & 0 \end{array} \qquad \sum \lambda_i \epsilon_i$$

splits, thus there is a map $g' : X' \to N$ with $g'gu = \sum \lambda_i \pi_i$. Now, we can assume that $\lambda_1 = 1$. If we denote by $u_1 : N \to N^n \oplus N'$ the first inclusion, so that $\pi_1 u_1 = 1$ and $\pi_i u_1 = 0$ for $i \geqslant 2$, then $g'guu_1 = \sum \lambda_i \pi_i u_1 = 1_N$. This shows that uu_1 is a split monomorphism. $\square$

Recall that a module homomorphism $f : M \to N$ is said to be a *radical map* provided for any indecomposable direct summand M' of M with inclusion map u and any indecomposable direct summand N' of N with projection map p, the composite map pfu is not an isomorphism. The conclusion of the Ext-lemma means in particular that under the assumption $\dim \mathrm{Ext}^1(Q,N) < n$, the map u is not a radical map.

Boundedness lemma. *Let $\mathcal{N}$ be a finite set of indecomposable modules. Then there is a bound b such that any indecomposable module with a maximal submodule in* $\operatorname{add}\mathcal{N}$ *is of length at most b.*

Proof of the boundedness lemma. Let $\mathcal{N} = \{N_1, \dots, N_t\}$. For $1 \leqslant i \leqslant t$, let

$$e_i = \max\bigl\{\dim \operatorname{Ext}^1(S, N_i) \mid S \text{ a simple } \Lambda\text{-module}\bigr\}$$

(there are only finitely many simple Λ-modules, thus we can form the maximum). Let

$$b = 1 + \sum e_i |N_i|.$$

We claim that this is a bound we are looking for. Let M be indecomposable, and M' a maximal submodule of M which belongs to $\operatorname{add}\mathcal{N}$. Thus we can write $M' = \bigoplus_i N_i^{n_i}$ with natural numbers n_i. It follows that $|M| = 1 + \sum n_i |N_i|$. Now, assume $|M| > b$. Then there is an index i such that $n_i > e_i$, but then the Ext-lemma asserts that the inclusion map $M' \to M$ is not a radical map. However, this inclusion map is a proper monomorphism and M is indecomposable, a contradiction. □

Let $\mathcal{N}$ be any set of indecomposable modules, closed under cogeneration (this means that $\operatorname{add}\mathcal{N}$ is closed under submodules). Any module M has a maximal factor module $f^{\mathcal{N}}M$ which belongs to $\operatorname{add}\mathcal{N}$, namely $f^{\mathcal{N}}M = M/M'$, where M' is the intersection of the kernels of all the maps $M \to N$ with N in $\operatorname{add}\mathcal{N}$. Note that any map $f : M \to N$ with N in $\operatorname{add}\mathcal{N}$ factors through the projection $M \to M/M' = f^{\mathcal{N}}M$.

We call a Λ-module M $\mathcal{N}$*-critical* provided M does not belong to $\operatorname{add}\mathcal{N}$, but any proper submodule of M belongs to $\operatorname{add}\mathcal{N}$. Of course, any $\mathcal{N}$-critical module is indecomposable. We denote by $\kappa(\mathcal{N})$ the class of all $\mathcal{N}$-critical modules.

The boundedness lemma shows: *If $\mathcal{N}$ is a finite set of indecomposable modules, then the modules in $\kappa(\mathcal{N})$ are of bounded length.*

The following lemma deals with the essential construction in Roiter's proof of the first Brauer–Thrall conjecture [9]:

Coamalgamation lemma. *Let $\mathcal{N}$ be a set of indecomposable modules, closed under cogeneration. Take a set $M_1, \dots, M_m$ of indecomposable modules of equal length, which belong to $\kappa(\mathcal{N})$. Assume that there exists a non-zero module Q in* $\operatorname{add}\mathcal{N}$ *such that $f^{\mathcal{N}}M_i$ is isomorphic to Q for all i, say with epimorphism $q_i : M_i \to Q$. Let $M = \bigoplus_i M_i$ and consider the kernel K of the map $q = [q_1, \dots, q_m] : M \to Q$. Then, the inclusion map $f : K \to M$ is a radical map.*

Proof. (Following [9].) We write

$$f = \begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix} : K \to M = \bigoplus_{i=1}^{m} M_i,$$

where $f_i : K \to M_i$. In order to show that f is not a radical map, we have to rule out the existence of a map $u : M_i \to K$ with $f_i u = 1$. Assume such a map exists for some i. Without loss of generality, we can assume $i = 1$.

Consider the maps $f_i u : M_1 \to M_i$ for $i \geqslant 2$. Since M_1 and M_i are not isomorphic, the map $f_i u$ cannot be an isomorphism. Since M_1 and M_i are of equal length, the image of $f_i u$ is a proper submodule of M_i, thus in $\operatorname{add}\mathcal{N}$. Thus $f_i u = g_i q_1$, for some map $g_i : Q \to M_i$.

The exact sequence

$$0 \to K \xrightarrow{f} M = \bigoplus_{i=1}^{m} M_i \xrightarrow{q} Q \to 0$$

yields an anticommutative square

$$\begin{tikzcd}
& M_1 \arrow[dr, "q_1"] & \\
K \arrow[ur, "f_1"] \arrow[dr, "f'"'] & & Q \\
& M' \arrow[ur, "q'"'] &
\end{tikzcd}$$

where

$$M' = \bigoplus_{i \geqslant 2} M_i \quad \text{and} \quad f' = \begin{bmatrix} f_2 \\ \vdots \\ f_m \end{bmatrix}$$

and $q' = [q_2, \ldots, q_m]$. We also define

$$g' = \begin{bmatrix} g_2 \\ \vdots \\ g_m \end{bmatrix},$$

thus $g' q_1 = f' u$. We add the map u to the picture:

$$\begin{tikzcd}
& & M_1 \arrow[dr, "q_1"] & \\
M_1 \arrow[r, "u"] \arrow[urr, bend left, "1"] \arrow[dr, "q_1"'] & K \arrow[ur, "f_1"] \arrow[dr, "f'"] & & Q \\
& Q \arrow[r, "g'"'] & M' \arrow[ur, "q'"'] &
\end{tikzcd}$$

We see:

$$q_1 = q_1 f_1 u = -q' f' u = -q' g' q_1,$$

thus, since q_1 is surjective, $q' g' = -1$. In particular, q' is split epi, thus Q is isomorphic to a direct summand of M'. However, Q is a non-zero module in $\operatorname{add}\mathcal{N}$, whereas no indecomposable direct summand of M' belongs to $\mathcal{N}$. This contradiction completes the proof. $\square$

Proof of the take-off part of Theorem 2. We let $I_1 = \{1\}$, so that $\mathcal{A}(I_1)$ is just the class of all simple modules.

We use induction. Assume we have already constructed Gabriel–Roiter measures $I_1 < I_2 < \cdots < I_t$ for Λ of finite type, such that any other Gabriel–Roiter measure I for Λ satisfies $I_t < I$.

Let $\mathcal{N} = \bigcup_{i=1}^{t} \mathcal{A}(I_i)$, this is a finite class of indecomposable modules (up to isomorphism). Let us assume that not all indecomposable modules belong to $\mathcal{N}$ (this assumption is always satisfied in case Λ is of infinite representation type).

We form $\kappa(\mathcal{N})$. Note that any indecomposable Λ-module M which does not belong to $\mathcal{N}$ contains an indecomposable submodule which belongs to $\kappa(\mathcal{N})$ (namely, just take any indecomposable submodule of M which does not belong to $\mathcal{N}$ and which is minimal with this property). In particular, we see that $\kappa(\mathcal{N})$ is not empty. According to the boundedness lemma, the modules in $\kappa(\mathcal{N})$ are of bounded length, thus there are only finitely many possible Gabriel–Roiter measures for $\kappa(\mathcal{N})$. Let I_{t+1} be the smallest Gabriel–Roiter measure a module in $\kappa(\mathcal{N})$ can have. Since the indecomposable modules with Gabriel–Roiter measure I_{t+1} do not belong to $\mathcal{N}$, we see that $I_t < I_{t+1}$.

If M is an arbitrary indecomposable module which does not belong to $\mathcal{N}$, then, as we know, M has an indecomposable submodule M' which belongs to $\kappa(\mathcal{N})$. But then $I_{t+1} \leqslant \mu(M') \leqslant \mu(M)$. This shows that any Gabriel–Roiter measure I different from $I_1, I_2, \ldots, I_{t+1}$ satisfies $I_{t+1} < I$.

It remains to be shown that $\mathcal{A}(I_{t+1})$ is of finite type. First of all, we note that $\mathcal{A}(I_{t+1}) \subseteq \kappa(\mathcal{N})$. Namely, let M be indecomposable and $\mu(M) = I_{t+1}$. As we know, there is an indecomposable submodule M' of M which belongs to $\kappa(\mathcal{N})$. But then $I_{t+1} \leqslant \mu(M') \leqslant \mu(M) = I_{t+1}$ shows that $\mu(M') = \mu(M)$, thus $M' = M$ and therefore M belongs to $\kappa(\mathcal{N})$.

Since the modules in $\mathcal{A}(I_{t+1})$ have the same Gabriel–Roiter measure, they have the same length, say l. Fix a module $Q \neq 0$ in $\operatorname{add}\mathcal{N}$ of length smaller l and let $\mathcal{A}(I_{t+1}, Q)$ be the class of all modules M_i in $\mathcal{A}(I_{t+1})$ with $f^{\mathcal{N}} M_i$ isomorphic to Q. We are going to show that $\mathcal{A}(I_{t+1}, Q)$ contains only finitely many isomorphism classes of modules. Take pairwise non-isomorphic modules $M_1, \ldots, M_m$ in $\mathcal{A}(I_{t+1}, Q)$. As we have shown, they belong to $\kappa(\mathcal{N})$ so that all the assumptions of the coamalgamation lemma are satisfied. Thus, choose an epimorphism $q_i : M_i \to Q$ and form the kernel $f : K \to M$ of the map $q = [q_1, \ldots, q_m] : M = \bigoplus_i M_i \to Q$. The coamalgamation lemma asserts that f is a radical map. Since all the modules M_i belong to $\mathcal{A}(I_{t+1})$, the submodule K of M belongs to $\operatorname{add}\mathcal{A}(\leqslant I_{t+1})$, according to the main property (a). If any direct summand K' of K would belong to $\mathcal{A}(I_{t+1})$, then the corresponding embedding $K' \to M$ would be a split monomorphism by the main property (b), but this is impossible, since f is a radical map. This shows that K belongs to $\operatorname{add}\mathcal{N}$. If we denote by $N_1, \ldots, N_s$ the indecomposable modules in $\mathcal{N}$, one from each isomorphism class, then, up to isomorphism, we can write $K = \bigoplus N_i^{n_i}$. Let $e_i = \dim \operatorname{Ext}^1(Q, N_i)$. According to the Ext-lemma, we must have $n_i \leqslant e_i$, since the inclusion map f is a radical map. This shows that $|M| \leqslant |Q| + \sum e_i |N_i|$. On the other hand, $|M| = |\bigoplus_{i=1}^{m} M_i| = m \cdot l$. Altogether, we see that

$$m \leqslant \frac{1}{l}\Big(|Q| + \sum e_i |N_i|\Big).$$

This shows that $\mathcal{A}(I_{t+1}, Q)$ contains only finitely many isomorphism classes.

Now, for every module M_i in $\mathcal{A}(I_{t+1})$, the factor module $f^{\mathcal{N}} M_i$ is non-zero, has length smaller than l and belongs to $\operatorname{add}\mathcal{N}$. Since there are only finitely many isomorphism classes Q of non-zero modules in $\operatorname{add}\mathcal{N}$ of length smaller than l, it follows that $\mathcal{A}(I_{t+1})$ itself contains only finitely many isomorphism classes. $\square$

Proof of Theorem 3. Consider the GR-inclusion graph for the modules in the various classes $\mathcal{A}(I_i)$: its vertices are the isomorphism classes of these modules (we will denote the isomorphism class of M by $[M]$) and there is an arrow $[N] \to [M]$ provided there is a Gabriel–Roiter inclusion $N \to M$. Obviously the only sources of this graph are the simple modules, thus there are only finitely many sources.

We are going to show that every vertex $[N]$ has only finitely many successors. Namely, if $N \to M$ is a Gabriel–Roiter inclusion, and N belongs to $\mathcal{A}(I_t)$, then every proper indecomposable submodule belongs to $\mathcal{A}(\leqslant I_t)$. The boundedness lemma asserts that there is a bound b such that any indecomposable module with all proper submodules in $\operatorname{add}\mathcal{A}(\leqslant I_t)$ is of length at most b. Thus $|M| \leqslant b$. But there are only finitely many Gabriel–Roiter measures $I_i \subseteq [1, b]$. Since any class $\mathcal{A}(I_i)$ contains only finitely many isomorphism classes, it follows that there are only finitely many possibilities for $[M]$.

Since we deal with an infinite, but locally finite graph with finitely many sources, König's graph theorem asserts that there exists an infinite path. $\square$

4. The landing part

Construction of the measures I^i. We denote by $[1, t]$ the set of natural numbers i with $1 \leqslant i \leqslant t$. Let $I^1 = [1, t]$, where t is the maximal length of an indecomposable injective module. Note that a module N with Gabriel–Roiter measure $[1, t]$ is indecomposable injective and of maximal length (namely, it has simple socle, thus its injective envelope also has simple socle and therefore has Gabriel–Roiter measure $[1, t']$ with $t' \geqslant t$). This shows that $[1, t]$ is the maximal possible Gabriel–Roiter measure and that $\mathcal{A}([1, t])$ is the class of all indecomposable injective modules of length t, in particular $\mathcal{A}([1, t])$ is a finite class of isomorphism classes.

Now assume we have constructed already Gabriel–Roiter measures I^i for Λ with $1 \leqslant i \leqslant m$, such that

(1) $I^1 > I^2 > \cdots > I^m$,
(2) that $I^m > I$ for any other Gabriel–Roiter measure I for Λ, and
(3) that any I^i is of finite type, for $1 \leqslant i \leqslant m$.

Let $\mathcal{X}$ be the additive category with indecomposable modules in the various $\mathcal{A}(I^i)$, with $1 \leqslant i \leqslant m$, and $\mathcal{X}'$ the additive subcategory of all modules with no indecomposable direct summand in these $\mathcal{A}(I^i)$. Note that since $\mathcal{X}$ is finite, any module in $\mathcal{A}$ has an $\mathcal{X}'$-cover [3, 4.1]. Let M be an additive generator of $\mathcal{X}$ and let Q be an injective cogenerator of $\mathcal{A}$. Let $g : M' \to M \oplus Q$ be an $\mathcal{X}'$-cover. Now, let I^{m+1} be the maximal Gabriel–Roiter measure of the indecomposable direct summands of M'. If N is any indecomposable module in $\mathcal{X}'$, then N is cogenerated by $M \oplus Q$. Since g is an $\mathcal{X}'$-cover, and N is in $\mathcal{X}'$, we see that N is cogenerated by M'. The main property implies that $\mu(N)$ is bounded by the maximum of the Gabriel–Roiter measures of the indecomposable direct summands of M', thus by I^{m+1}. Also, if $\mu(N) = I^{m+1}$, then N actually is isomorphic to a direct summand of M'. This shows that all the modules in $\mathcal{A}(I^{m+1})$ are indecomposable direct summands of M', in particular $\mathcal{A}(I^{m+1})$ is finite. This completes the induction.

Remark. The proof can be modified in order to show that all the modules in the various $\mathcal{A}(I^i)$ are preinjective. However, this also can be derived from Theorem 4 and the results in [3], as we want to outline now. Let $\mathcal{C}$ be any full subcategory $\mathcal{C}$ of a fixed abelian category. Auslander–Smalø denote by $\mathbf{I}(\mathcal{C})$ the class of all modules I in $\mathcal{C}$ such that any monomorphism $I \to C$ with C an arbitrary object in $\mathcal{C}$ splits. Given our category $\mathcal{A}$, they write $\mathbf{I}_0$ for $\mathbf{I}(\mathcal{A})$, and inductively $\mathbf{I}_i = \mathbf{I}(\mathcal{A}_i)$, where $\mathcal{A}_i$ is obtained from $\mathcal{A}$ by deleting all the modules which have an indecomposable direct summand in $\mathbf{I}_0, \ldots, \mathbf{I}_{i-1}$. Finally, $\mathbf{I}_\infty$ is obtained from $\mathcal{A}$ by deleting all the modules which have an indecomposable direct summand in any $\mathbf{I}_i$, with $i \in \mathbb{N}_0$. The modules in $\bigoplus_{i\in\mathbb{N}_0} \mathbf{I}_i$ are said to be *preinjective*.

Proposition. *We have* $\mathcal{A}(I^t) \subseteq \mathbf{I}_0 \cup \cdots \cup \mathbf{I}_{t-1}$.

Proof. (By induction in t.) Clear for $t = 0$, since the modules in $\mathcal{A}(I^1)$ are the indecomposable injective modules of largest length (in particular, all are injective). Now assume, we know by induction that

$$\mathcal{A}(I^i) \subseteq \mathbf{I}_0 \cup \cdots \cup \mathbf{I}_{i-1},$$

for all $1 \leqslant i \leqslant t$. As a consequence, we have

$$\mathcal{A}(I^1) \cup \cdots \cup \mathcal{A}(I^t) \subseteq \mathbf{I}_0 \cup \cdots \cup \mathbf{I}_{t-1}.$$

This implies that

$$\bigcup_{i\geqslant t} \mathbf{I}_i \subseteq \mathcal{A}(\leqslant I^{t+1})$$

(where the index $i \geqslant t$ should include also $i = \infty$). It follows that

$$\bigcup_{i\geqslant t} \mathbf{I}_i = \mathcal{A}(\leqslant I^{t+1}) \setminus \mathbf{I}_0 \cup \cdots \cup \mathbf{I}_{t-1}.$$

Now by the splitting part of the main property, we know that

$$\mathcal{A}(I^{t+1}) \subseteq \mathbf{I}\big(\mathcal{A}(I^{t+1})\big),$$

thus

$$\mathcal{A}(I^{t+1}) \setminus \mathbf{I}_0 \cup \cdots \cup \mathbf{I}_{t-1} \subseteq \mathbf{I}\big(\mathcal{A}(I^{t+1}) \setminus \mathbf{I}_0 \cup \cdots \cup \mathbf{I}_{t-1}\big) = \mathbf{I}\Bigg(\bigcup_{i\geqslant t} \mathbf{I}_i\Bigg) = \mathbf{I}_t.$$

This shows that

$$\mathcal{A}(I^{t+1}) \subseteq \mathbf{I}_0 \cup \cdots \cup \mathbf{I}_t.$$

Acknowledgements

This is a slightly polished version of a paper which has been finished in May 2003 and the author is grateful to Philipp Fahr, Masahisa Sato and the referee for spotting misprints and inaccuracies. The paper is essentially self-contained, as basic reference one may consult the book of Auslander, Reiten and Smalø [2]. As we have mentioned, some assertions are well known, but there do not seem to be well-distributed references; for the convenience of the reader, the corresponding proofs have been included. In particular, this concerns Roiter's coamalgamation lemma. For further results concerning the Gabriel–Roiter measure, we refer to the forthcoming lectures notes [8]. In these notes, the relationship between the Gabriel–Roiter measure and Auslander–Reiten theory will be made explicit (for any Gabriel–Roiter inclusion $N \subset M$, the factor module M/N turns out to be an epimorphic image of the Auslander–Reiten translate $\tau^{-1}N$). This provides alternative proofs of some results of the present paper. But let us stress that the Gabriel–Roiter measure has to be considered as being more elementary than the Auslander–Reiten invariants, since the latter rely on the complete module category and not only on the individual module. For this reason, we feel that the direct approach of the present paper is of interest in its own.

Appendix A. A third proof of the first Brauer–Thrall conjecture

Theorem 2 yields two different proofs for Brauer–Thrall I: Since any infinite set of Gabriel–Roiter measures for Λ provides indecomposable modules of arbitrarily large length, both the take-off part as well as the landing part contain indecomposable modules of arbitrarily large length. The considerations concerning the take-off part rely on the coamalgamation lemma. Here we want to outline that the coamalgamation lemma can be used in a slightly different way in order to obtain a third proof for Brauer–Thrall I.

Proposition. *Assume that $\mathcal{A}(I)$ contains an infinite set $\mathcal{M}$ of pairwise non-isomorphic modules. Then there do exist arbitrarily large indecomposable modules which are cogenerated by $\mathcal{M}$.*

Proof. Let $\mathcal{N}$ be the class of all indecomposable modules M which are either simple or else cogenerated by $\mathcal{M}$ and not belonging to $\mathcal{A}(I)$.

Assume the length of the modules in $\mathcal{N}$ is bounded by b. In particular, only finitely many Gabriel–Roiter measures I' occur for the modules in $\mathcal{N}$. Using induction on the number of such Gabriel–Roiter measures, we see that we can assume that for any such Gabriel–Roiter measure I', there are only finitely many isomorphism classes in $\mathcal{A}(I') \cap \mathcal{N}$ (namely, if there is $I' < I$ such that $\mathcal{A}(I') \cap \mathcal{N}$ contains infinitely many isomorphism classes, then by induction there are arbitrarily large indecomposable modules which are cogenerated by $\mathcal{A}(I') \cap \mathcal{N}$, but these modules are also cogenerated by $\mathcal{M}$). Thus $\mathcal{N}$ is a finite set of isomorphism classes.

Note that the class $\mathcal{N}$ is closed under cogeneration, thus for any module M, we may consider its maximal factor module $f^{\mathcal{N}}M$ which belongs to add$\mathcal{N}$. Since $\mathcal{N}$ contains all the simple modules, $f^{\mathcal{N}}M$ is non-zero, for non-zero M.

Let l be the length of the modules in $\mathcal{A}(I)$. There are only finitely many isomorphism classes of modules Q in $\operatorname{add}\mathcal{N}$ of length smaller l, but infinitely many isomorphism classes $M \in \mathcal{M}$, thus there are infinitely many isomorphism classes M_i of modules in $\mathcal{M}$ with $f^{\mathcal{N}} M_i$ isomorphic to a fixed module Q.

Take pairwise non-isomorphic modules $M_1, \ldots, M_m$ in $\mathcal{M}$ with epimorphisms $q_i : M_i \to Q = f^{\mathcal{N}} M_i$. Of course, all the modules in $\mathcal{M}$ belong to $\kappa(\mathcal{N})$, thus all the assumptions of the coamalgamation lemma are satisfied. This means: consider the map $q = [q_1, \ldots, q_m] : M = \bigoplus_i M_i \to Q$ and form the kernel $f : K \to M$, then f is a radical map. Since all the modules M_i belong to $\mathcal{M}$, the submodule K of M is cogenerated by $\mathcal{M}$. Since f is a radical map, the main property (b) asserts that no direct summand of K can belong to $\mathcal{A}(I)$, thus K belongs to $\operatorname{add}\mathcal{N}$. Now, as in the proof of Theorem 2 (take-off part) , let $N_1, \ldots, N_s$ be the indecomposable modules in $\mathcal{N}$, one from each isomorphism class, write $K = \bigoplus N_i^{n_i}$. Let $e_i = \dim \operatorname{Ext}^1(Q, N_i)$, so that $n_i \leqslant e_i$. This shows that $m \leqslant |\bigoplus_{i=1}^m M_i| = |M| \leqslant |Q| + \sum e_i |N_i|$, thus m is bounded, a contradiction.

This contradiction shows that the length of the modules in $\mathcal{N}$ cannot be bounded and completes the proof. $\square$

Remark 1. It should be noted that the indecomposable modules obtained by the proposition usually will not belong to the take-off part (and none of them will belong to the landing part). For example, if Λ is a tubular algebra (see [7]), there are non-sincere dimension vectors $\mathbf{d}$ and $\mathbf{d}'$ which are not multiples of each other, which occur as dimension vectors of one-parameter families of indecomposable modules. One of these families will cogenerate only finitely many indecomposable modules from the take-off part.

Remark 2. The coamalgamation lemma provides exact sequences of the form

$$0 \to K \xrightarrow{f} M = \bigoplus_{i=1}^{m} M_i \xrightarrow{q} Q \to 0$$

and in the application above, we have seen that K has large indecomposable direct summands. One may wonder whether in such a situation K has to be indecomposable. As a typical example, consider modules M_i which belong to the primitive one-parameter family of the four-subspace quiver (thus any M_i is indecomposable and of length 6), let $\mathcal{N}$ be the class of all simple modules (so that the epimorphism $q_i : M_i \to Q$ has as kernel the radical of M_i). An easy calculation shows that in this case K is always the direct sum of *two* indecomposable modules.

Appendix B. Visualization and examples

We want to provide a more intuitive understanding of the Gabriel–Roiter measure, in particular of the total ordering on the set of Gabriel–Roiter measures for an artin algebra Λ. Let $\mathcal{P}_l(\mathbb{N}_1)$ be the set of all subsets I of $\mathbb{N}_1$ such that for any $n \in \mathbb{N}_1$, there is $n' \geqslant n$ with $n' \notin I$.

Lemma B.1. *The Gabriel–Roiter measure $\mu(M)$ of any module M belongs to $\mathcal{P}_l(\mathbb{N}_1)$.*

Proof. There is $m \in \mathbb{N}_1$ such that any indecomposable injective Λ-module has length at most m. Let $\mu(M) = \{a_1 < a_2 < \cdots < a_i < \cdots\}$ and assume that for some n we have $a_{n+t} = a_n + t$ for all $t \in \mathbb{N}_1$. Let $s = m \cdot a_n$.

There is a chain of indecomposable submodules $M_1 \subset M_2 \subset \cdots \subset M_{n+s}$ with $|M_i| = a_i$ for $1 \leqslant i \leqslant n+s$. Since $|M_{n+t}| = a_{n+t} = a_{n+t-1} + 1 = |M_{n+t-1}| + 1$, we see that M_{n+t-1} is a maximal submodule of M_{n+t}. Since M_{n+t} is indecomposable, the socle of M_{n+t} has to be contained in M_{n+t-1}. Inductively, we see that the socle of M_{n+t} is contained in M_n, for any $t \geqslant 1$, in particular, the socle of M_{n+s} is contained in M_n, thus M_{n+s} can be embedded into the injective envelope of M_n. Since any indecomposable injective module is of length at most m, the injective envelope of M_n has length at most $m \cdot a_n$, thus $|M_{n+s}| \leqslant m \cdot a_n$. But $|M_{n+s}| = |M_n| + s = (m+1)a_n > m \cdot a_n$, a contradiction. $\square$

One may embed $\mathcal{P}_l(\mathbb{N}_1)$ into the real interval [0, 1] as follows:

Lemma B.2. *The map $r : \mathcal{P}_l(\mathbb{N}_1) \to \mathbb{R}$ given by $r(I) = \sum_{i \in I, i \geqslant 2} \frac{1}{2^{i-1}}$ for $I \in \mathcal{P}_l(\mathbb{N}_1)$ is injective, its image is contained in the interval $[0,1]$ and it preserves and reflects the ordering.*

Proof. The essential consideration is the following: Let I, J belong to $\mathcal{P}_l(\mathbb{N}_1)$ with $I < J$. Then $r(I) = r(I \cap J) + r(I \setminus J)$ and $r(J) = r(I \cap J) + r(J \setminus I)$. Let a be the smallest element in $J \setminus I$. Then $r(J \setminus I) \geqslant \frac{1}{2^{a-1}} = \sum_{i > a} \frac{1}{2^{i-1}} > r(I \setminus J)$, since $I \setminus J$ is a proper subset of $\{i \in \mathbb{N}_1 \mid i > a\}$. $\square$

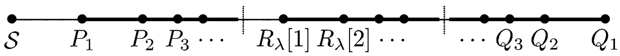
Remark 1. The map r can be defined not just on $\mathcal{P}_l(\mathbb{N}_1)$, but on all of $\mathcal{P}(\mathbb{N}_1)$, however it will no longer be injective (indeed, for any element I in $\mathcal{P}(\mathbb{N}_1) \setminus \mathcal{P}_l(\mathbb{N}_1)$, there is a unique finite set I' with $r(I) = r(I')$). Of course, one easily may change the definition of r in order to be able to embed all of $\mathcal{P}(\mathbb{N}_1)$ into $\mathbb{R}$: just use say 3 instead of 2. However, our interest lies in the Gabriel–Roiter measures which occur for finite dimensional algebras and Lemma B.1 assures us that the definition of r as proposed is sufficient for these considerations.

Remark 2. One may wonder why we use here the map r, and not just $r' : \mathcal{P}_l(\mathbb{N}_1) \to \mathbb{R}$ with $r'(I) = \sum_{i \in I} \frac{1}{2^i}$. The main reason is that the map r attaches 0 to the simple modules; in this way the rhombic picture defined below fits quite well into one of the quadrants of a coordinate system. Future investigations should decide which choice is preferable.

Example 1 (*The Kronecker quiver $\widetilde{A}_{11}$*). This is the path algebra $k\Delta$ of the quiver Δ with vertices a, b and two arrows $b \to a$; its representations are called *Kronecker modules*. There are two simple Kronecker modules, the projective simple module $S(a)$ and the injective simple module $S(b)$. If M is a Kronecker module, its *dimension vector* is of the form $\mathbf{dim}\, M = (d_a, d_b)$, where d_a is the Jordan–Hölder multiplicity of $S(a)$, and d_b that of $S(b)$. The dimension vectors of the indecomposable modules are of the form (x, y) with $|x - y| \leqslant 1$. Here is the complete list of the indecomposable representations in case k is algebraically closed:

- The preprojective modules P_n for $n \in \mathbb{N}_0$, with $\mathbf{dim}\, P_n = (n+1, n)$ and $\mu(P_n) = \{1, 3, 5, \ldots, 2n+1\}$.
- The preinjective modules Q_n for $n \in \mathbb{N}_0$, with $\mathbf{dim}\, Q_n = (n, n+1)$ and $\mu(Q_n) = \{1, 2, 4, 6, \ldots, 2n, 2n+1\}$.
- The regular modules $R_\lambda[n]$ for $\lambda \in \mathbb{P}^1(k)$ and $n \in \mathbb{N}_1$, with $\mathbf{dim}\, R_\lambda[n] = (n, n)$ and $\mu(R_\lambda[n]) = \{1, 2, 4, 6, \ldots, 2n\}$.

The totally ordered set of all the Gabriel–Roiter measures for the Kronecker quiver looks as follows:



Here $\mathcal{S} = \mathcal{A}(\{1\}) = \{S(a), S(b)\}$. Note that there are precisely two accumulation points, indicated by the dotted vertical lines, they correspond to the only two Gabriel–Roiter measures for infinitely generated modules: to the left, there is $\{1, 3, 5, 7, \ldots\}$, this is the Gabriel–Roiter measure for all indecomposable torsionfree modules; to the right, there is $\{1, 2, 4, 6, 8, \ldots\}$, this is the Gabriel–Roiter measure for the so-called Prüfer modules (an account of the structure theory for infinitely generated Kronecker modules can be found for example in [6]).

In case k is not algebraically closed, we have to take into account field extensions of k, or better indecomposable $k[T]$-module of finite length N, where $k[T]$ is the polynomial ring over k in one variable T. Any indecomposable $k[T]$-module N of length n and with a simple submodule of dimension d gives rise to a regular Kronecker module with dimension vector (nd, nd) and Gabriel–Roiter measure $\{1, 3, 5, \ldots, 2d-1, 2d; 4d, 6d, \ldots, 2nd\}$. Thus we see that *the Gabriel–Roiter measure for the path algebra $k\Delta$ of a quiver Δ may depend on k* (and usually will).

Example 2 (*The tame hereditary algebra of type $\widetilde{A}_{21}$*). This algebra shows that even for a tame hereditary algebra infinitely many preinjective modules may be outside the landing part. We also encounter some other features which one should be aware of.

We deal with the path algebra of the following quiver

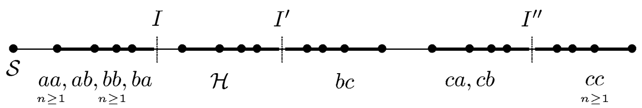
$$\begin{tikzcd} & b \arrow[dl] & \\ a & & c \arrow[ul] \arrow[ll] \end{tikzcd}$$

In order to list all the indecomposable Λ-modules, we use that Λ is a string algebra. Thus the indecomposable modules are string or band modules. Again, we restrict to the case of k being algebraically closed.

- There is a unique one-parameter family of band modules; they are of the form $R_\lambda[n]$, where $\lambda \in k \setminus \{0\}$ and $n \in \mathbb{N}_1$, with Gabriel–Roiter measure $\mu(R_\lambda) = \{1, 2, 3; 6, 9, \ldots, 3n\}$.

In order to write down the string modules, we use word in $\alpha, \beta, \gamma^{-1}$; the relevant distinction is given by fixing the vertices x, y such that the word starts in x and ends in y (always $n \in \mathbb{N}_0$):

xy	Properties	Dimension	GR-measure
• aa	preprojective	$3n+1$	$1, 2, 4, 5, 7, 8, \ldots, 3n-2, 3n-1, 3n+1$
• ab	preprojective	$3n+2$	$1, 2, 4, 5, 7, 8, \ldots, 3n-2, 3n-1, 3n+1, 3n+2$
• ac	homogeneous	$3n+3$	$1, 2, 3; 6, 9, \ldots, 3n$
• ba	regular, non-homog.	$3n+3$	$1, 2, 4, 5, 7, 8, \ldots, 3n-2, 3n-1, 3n+1, 3n+3$
• bb	regular, non-homog.	$3n+1$	$1, 2, 4, 5, 7, 8, \ldots, 3n-2, 3n-1, 3n+1$
• bc	preinjective	$3n+2$	$1, 2, 3; 6, 9, \ldots, 3n; 3n+2$
• ca	regular, non-homog.	$3n+2$	$1, 2, 3; 5, 6, 8, 9, \ldots, 3n-1, 3n, 3n+2$
• cb	regular, non-homog.	$3n+3$	$1, 2, 3; 5, 6, 8, 9, \ldots, 3n+2, 3n+3$
• cc	preinjective	$3n+1$	$1, 2, 3; 5, 6, 8, 9, \ldots, 3n-1, 3n; 3n+1$



Here, $\mathcal{H}$ denotes the class of all homogeneous modules (the bands as well as the strings of type ac), whereas $\mathcal{S}$ are the simple modules.

Some observations:

(1) There are many "*maximal*" GR-measures I (maximality should mean that no other GR-measure starts with I), in particular see ba, but also bc and cc.
(2) The *take-off part* contains all the preprojective modules, but in addition also half of the non-homogeneous tube (namely all the regular modules which have the simple module $S(b)$ as submodule.
(3) The *landing part* contains only half of the preinjective modules (the modules bc are preinjective).
(4) The GR-measure apparently does not distinguish modules which have quite different behaviour, see aa and bb (however, aa and bb will be distinguished in case we invoke the dual concepts, see the next appendix).
(5) There are three *accumulation points* I, I', I'':

$$I = \{1, 2, 4, 5, 7, 8, 10, 11, \ldots\},$$
$$I' = \{1, 2, 3, 6, 9, 12, 15, \ldots\},$$
$$I'' = \{1, 2, 3, 5, 6, 8, 9, 11, 12, \ldots\}.$$

The first one I is the Gabriel–Roiter measure of the torsionfree modules; I' is the Gabriel–Roiter measure for all the Prüfer modules arising from homogeneous tubes; I'' is that of the Prüfer module containing the 2-dimensional indecomposable regular module as a submodule.
(6) There is one additional Prüfer module, it contains the simple module $S(b)$ as a submodule: this module does *not* have a Gabriel–Roiter filtration!

Appendix C. Dualization and the rhombic picture

Dualization. Almost all the considerations presented above can be dualized and then they yield corresponding dual results. This means that instead of looking at filtrations

$$0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$$

with M_i indecomposable for $1 \leqslant i \leqslant t$, we now look at such filtrations with M/M_{i-1} indecomposable for $1 \leqslant i \leqslant t$. We prefer to use now the opposite order on $\mathcal{P}(\mathbb{N}_1)$, we denote it by $\leqslant^*$ (and $<^*$), thus $I \leqslant^* J$ iff $J \leqslant I$. For a (not necessarily finitely generated) Λ-module M, let $\mu^*(M)$ be the infimum of the sets $\{|M/M_1|, \ldots, |M/M_t|\}$ in $(\mathcal{P}(\mathbb{N}_1), \leqslant^*)$, where $M_1 \subset M_2 \subset \cdots \subset M_t$ is a chain of submodules of M with M/M_{i-1} indecomposable for $1 \leqslant i \leqslant t$, we call $\mu^*(M)$ the *Gabriel–Roiter comeasure* of M. We say that J is a Gabriel–Roiter comeasure for Λ provided there exists an indecomposable module M with $\mu^*(M) = J$.

In order to visualize $(\mathcal{P}_l(\mathbb{N}_1), \leqslant^*)$, we use the embedding $r^*: (\mathcal{P}_l(\mathbb{N}_1), \leqslant^*) \to \mathbb{Q}$ given by $r^*(I) = -r(I)$. Note that for any non-zero module M, we have $-1 < r^*(\mu^*(M)) \leqslant 0$.

The dual version of the main property reads as follows:

Main property*. *Let $Y_1, \ldots, Y_t, Z$ be indecomposable Λ-modules of finite length and assume that there is an epimorphism $g: \bigoplus_{i=1}^t Y_i \to Z$.*

(a) *Then* $\max \mu^*(Y_i) \leqslant^* \mu^*(Z)$.
(b) *If $\mu^*(Z) = \max \mu^*(Y_i)$, then g splits.*
(c) *If $\max \mu^*(Y_i)$ starts with $\mu^*(Z)$, then there is some j such that gu_j is surjective, where $u_j : Y_j \to \bigoplus_i Y_i$ is the canonical inclusion.*

As a consequence, we see that the class of modules which are direct sums of modules M with $I \leqslant^* \mu^*(M)$ for some set $I \subseteq \mathbb{N}_1$ is closed under factor modules. In this way, one obtains a second interesting filtration of the category of all Λ-modules by subcategories, now these subcategories are closed under factor modules.

Let us formulate the dual versions of Theorem 2 and Theorem 4 (there does not seem to exist a dual version of Theorem 3, since Theorem 3 deals with infinitely generated modules):

Theorem 2*. *Let Λ be of infinite representation type. Then there are Gabriel–Roiter comeasures J_t, J^t for Λ with*

$$J_1 <^* J_2 <^* J_3 <^* \cdots <^* J^3 <^* J^2 <^* J^1$$

such that any other Gabriel–Roiter comeasure J for Λ satisfies $J_t <^* J <^* J^t$ for all $t \in \mathbb{N}_1$, and all these Gabriel–Roiter comeasures J_t and J^t are of finite type.

(We do not have a suggestion how to call the modules in $\bigcup_t \mathcal{A}(J_t)$ or in $\bigcup_t \mathcal{A}(J^t)$. Maybe, $\bigcup_t \mathcal{A}(J_t)$ should be called the $*$-take-off part of the category $\mathcal{A}$, and $\bigcup_t \mathcal{A}(J^t)$ the $*$-landing part of $\mathcal{A}$; but $\bigcup_t \mathcal{A}(J^t)$ is really the dual (thus $*$) of the take-off part.) The indecomposable modules which belong neither to $\bigcup_t \mathcal{A}(J_t)$ nor to $\bigcup_t \mathcal{A}(J^t)$ may be said to be form the $*$-*central part*.

Note that for any n, there are only finitely many isomorphism classes of indecomposable modules of length n which belong to $\bigcup_t \mathcal{A}(J_t)$ or to $\bigcup_t \mathcal{A}(J^t)$.

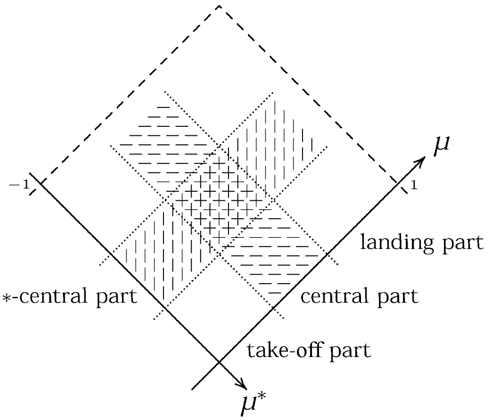
The modules in $\mathcal{A}(J^1)$ are just the simple modules, those in $\mathcal{A}(J^2)$ are the uniform modules of Loewy length 2 of largest possible length. On the other hand, the modules in $\mathcal{A}(J_1)$ are the indecomposable projective modules of largest possible length.

Theorem 4*. *The modules in $\bigcup_t \mathcal{A}(J_t)$ are preprojective.*

The rhombus. We are going to use now both the measure and the comeasure at the same time. Given a pair (J, I) of finite subsets I, J of $\mathbb{N}_1$, we may consider the module class

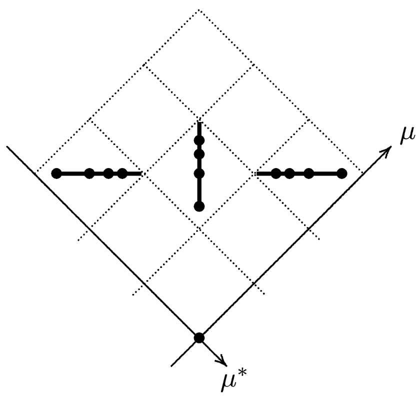
$$\mathcal{A}(J, I) = \big\{M \mid M \text{ indecomposable},\ \mu^*(M) = J,\ \mu(M) = I\big\},$$

thus we attach to a module M the pair $(\mu^*(M), \mu(M))$. The possible pairs (J, I) can be considered (via r^* and r) as elements in the rational plane $\mathbb{Q}^2$:



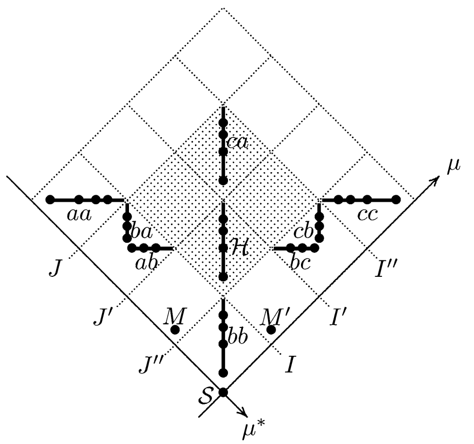
The horizontally dashed region is the central part (in between the take-off part and the landing part); the vertically dashed region is the $*$-central part. The main information one should keep in mind: *The only possible pairs (J, I) of finite subsets of $\mathbb{N}_1$ such that $\mathcal{A}(J, I)$ contains infinitely many isomorphism classes, are those which belong both to the central and the $*$-central part.*

Example 1. *The Kronecker quiver*, with k algebraically closed:



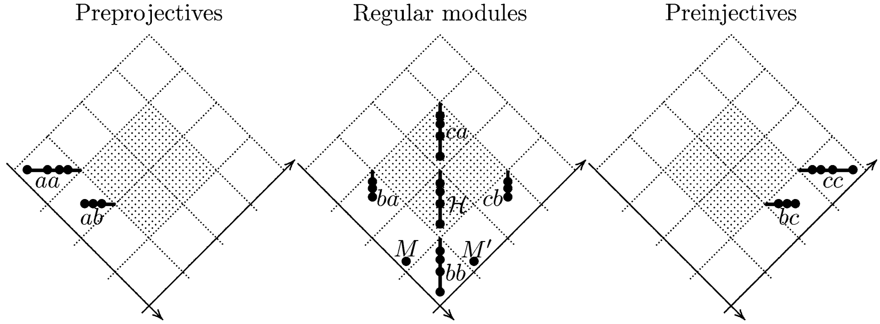
The picture obtained is nearly the same as the customary visualization, the only exception being the position of the simple modules. One should be aware that the commonly accepted visualization with the preprojective and the preinjective modules being drawn horizontally and the tubes being drawn vertically in the middle was based mainly on the feeling that this arrangement reflects much of the structure of the category, but for the actual position of the individual modules there was no further mathematical justification. The rhombic picture should be seen as a definite reassurance in this case (but it suggests deviations in other cases). Even for the Kronecker quiver, one should be aware that there does exist a deviation, namely the position of the simple modules. Of course, they are usually drawn far apart, one at the left end, the other at the right end, now they are located at the same position: in the middle lower corner. But note that the rhombic picture for the Kronecker quiver and the algebra $k[X,Y]/(X,Y)^2$ do not differ, and the usual Auslander–Reiten picture for the latter algebra puts its unique simple module precisely at this position (and bends down the preprojective modules on the left as well as the preinjective modules on the right to form half circles).

Example 2 (*The tame hereditary algebra of type* $\widetilde{A}_{21}$). Here is the rhombic picture, for k algebraically closed:



Two modules have to be specified separately, the indecomposable modules M, M' of length 3 and Loewy length 2: M is local, M' uniform; note that M has type ba, M' type cb. The accumulation points I, I', I'' for the Gabriel–Roiter measure are marked on the μ-axis; similarly, the accumulation points J, J', J'' for the Gabriel–Roiter comeasure are marked on the μ^*-axis (note that $J = I''$, $J' = I'$, $J'' = I$ in $\mathcal{P}(\mathbb{N}_1)$). The intersection of the central and the $*$-central part has been dotted, this region contains for every $n \in \mathbb{N}_1$ a $\mathbb{P}^1(k)$-family of indecomposable representations of length $3n$.

One immediately realizes that the rhombic picture again corresponds quite well to the commonly used visualization, at least after deleting the simple modules. The preprojective and the preinjective modules are arranged horizontally, the regular modules vertically (there is one exceptional tube of rank 2, it has four types of indecomposable modules, namely the types ca, ba (including M), bb and cd (including M'). Let us take apart these three parts of the category:



Appendix D. Take-off algebras

We call Λ a *take-off algebra* provided any indecomposable projective Λ-module belongs to its take-off part.

Lemma. *Let Λ be a representation-infinite artin algebra, and let A be the annihilator of the take-off part. Then Λ/A is a* (*not necessarily connected*) *representation-infinite take-off algebra.*

Recall that a module M is said to be *torsionless* if it can be embedded into a projective module. *If Λ is a take-off algebra, then there are only finitely many isomorphism classes of indecomposable torsionless modules.* Namely, there are only finitely many indecomposable projective modules. If they belong to the take-off part, they belong to $\mathcal{A}(\leqslant I_t)$ for some t; but then all indecomposable torsionless modules belong to $\mathcal{A}(\leqslant I_t)$.

Of course, the *annihilator A of the take-off part* is the set of all elements $\lambda \in \Lambda$ such that $\lambda M = 0$ for any module M in the take-off part of Λ. This is a two-sided ideal of Λ, and the direct sum $\bigoplus M$ of all indecomposable take-off modules (one from each isomorphism class) is a faithful Λ/A-module. Already a suitable finite direct sum of indecomposable take-off modules will be faithful as Λ/A-module, say the direct sum of the indecomposable modules in $\mathcal{A}(\leqslant I_t)$ for some $t \in \mathbb{N}_1$. But since $\mathcal{A}(\leqslant I_t)$ is closed under cogeneration, it follows that all the indecomposable projective Λ/A-modules belong to $\mathcal{A}(\leqslant I_t)$. Thus Λ/A is a take-off algebra.

In order to see that Λ/A is not necessarily connected, consider the path algebra of the quiver

$$a \longleftarrow b \leftleftarrows c$$

Here, A is generated by the arrow $a \leftarrow b$. This example shows also that an ideal A' of Λ such that Λ/A' is a take-off algebra, may be incomparable with respect to A: just let A' be the ideal generated by the double arrows.

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